

Thermodynamics and Phase Transition in Rotational Kiselev Black Hole

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ABSTRACT

We calculate the thermodynamical features of rotational Kiselev black holes, specifically we use one order approximate of horizon to calculate thermodynamical features for all ω . The thermodynamics features include areas, entropies, horizon radii, surface gravities, surface temperatures, Komar energies and irreducible masses at the Cauchy horizon and Event horizon. At the same time the products of these features have been discussed. We find that the products are independent with mass of black hole and determined by ω and α . The features in the situations of $\omega = -2/3, 1/3$ and 0 (quintessence matter, radiation and dust) have been discussed in detail. We also generalize the Smarr mass formula and Christodoulou-Ruffini mass formula to these black holes. Finally we study the phase transition for black holes with different ω and obtain the state equation. We analyze the phase transition for $\omega = 1/3$, and find that α shifts the critical point of phase transition.

Subject headings: Thermodynamics, Rotational Kiselev black hole, Quintessence matter, Radiation, Dust, Phase transition

1. INTRODUCTION

Black holes are important objects in physics especially for quantum gravity, they have the analogy of laws between dynamics and thermodynamics (Bardeen et al. (1973); Bekenstein (1972, 1974, 1973); Christodoulou (1970); Christodoulou & Ruffini (1971); Hawking (1974, 1971); Penrose & Floyd (1971)). The temperature and entropy are analogous to the surface gravity and area of black hole. The four laws of thermodynamics in black hole have

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been discussed (Akbar & Siddiqui (2007); Jamil & Akbar (2011); Jamil & Hussain (2011); Jamil et al. (2011)), and generalized in AdS spacetime (Wang et al. (2014)), Kerr-AdS black hole, and Kerr-Newman-AdS black holes. The spin entropy of rotating black hole have also been discussed (Curir (1979)). If the black hole exists Cauchy horizon and event horizon, we could study the products of these quantities, such as horizon and surface gravity etc.

The products of the thermodynamical features for black hole with rotational symmetry have been discussed (Majeed & Jamil (2015); Pradhan (2015a,b,c, 2014)). It is shown that sometimes the products of black hole are independent with ADM (Arnowitt-Deser-Misner) mass, but depend on charge and spin. Through studying these products, we could know these may turn to universal and hold for other solution for nontrivial surrounding black holes.

The phase transition in black hole has an interesting topics (Davies (1989, 1978, 1977); Guo et al. (2015); Hawking & Page (1982); Liao et al. (2016); Ruppeiner (2008); Saleh et al. (2011); Tharanath et al. (2014); Thomas et al. (2012)). For the Schwarzschild black hole the heat capacity is always negative and no phase transition exists. But for RN black hole, the phase transition will happen due to the existence of charge. If black hole has angular momentum and charge, the second order phase transition will happen. If black hole is surrounded by quintessence or other matters, the phase transition maybe happen.

For the quintessence matter (other energy matters) around schwarzschild black hole, the Einstein equation's solution have been obtained by Kiselev (Kiselev (2003)). These solution are determined by the parameters ω and α . Interestingly these solutions could also describe radiation or dust around black holes (Majeed et al. (2015)). Following these works, the particle collision have been discussed (Jamil et al. (2015)). The thermodynamical relation and phase transition have been obtained in Kiselev black hole for different situations (Majeed et al. (2015)). Recently the rotation Kiselev solution and Kerr-Newman kiselev solution have been obtained (Ghosh (2016); Oteev et al. (2016); Toshmatov et al. (2015); Xu & Wang (2016)). In this paper we want to study the thermodynamical relation in rotational Kiselev black hole, the products of thermal quantities, the generalized Smarr formula and Christodoulou-Ruffini mass formula. We also calculate the phase transition for these black holes in the case of $\omega = 1/3$.

The outline of this paper is as follows. In section 2, we discuss the basic aspects of rotational Kiselev black hole by one order perturbation for horizon. The results show that the products are determined by the parameters ω and α . We also generalize the Smarr mass formula and Christodoulou-Ruffini Mass formula in subsection 2.1. In section 3, the phase transition have been discussed, specially for radiation situation $\omega = 1/3$. The summary is presented in the last section. In this paper we set $G = \hbar = c = 1$.

2. THERMODYNAMICS IN KERR-LIAKE BLACK HOLE WITH QUINTESSENCE

The rotational symmetry solution for Einstein's equations in quintessence (or other energy matter) have been obtained by Toshmatov ([Toshmatov et al. \(2015\)](#))

$$ds^2 = -\left(1 - \frac{2Mr + \alpha r^{1-3\omega}}{\Sigma^2}\right)dt^2 + \frac{\Sigma^2}{\Delta_r}dr^2 - \frac{2a\sin^2\theta(2Mr + \alpha r^{1-3\omega})}{\Sigma^2}d\phi dt + \Sigma^2 d\theta^2 + \sin^2\theta(r^2 + a^2 + a^2\sin^2\theta \frac{2Mr + \alpha r^{1-3\omega}}{\Sigma^2})d\phi^2, \quad (1)$$

where

$$\Delta_r = r^2 - 2Mr + a^2 - \alpha r^{1-3\omega} \quad \Sigma^2 = r^2 + a^2 \cos^2\theta. \quad (2)$$

In this spacetime metric, M and a are the mass and spin of the black hole. α is the quintessence (or other matters) parameter. The state parameter ω is given by the equation of state $p = \omega\rho$. If the energy matter is quintessence, ω will be $-1 < \omega < -1/3$ to lead the universe acceleration. The case of $\omega = 1/3$ represents radiation, and the case of $\omega = 0$ represents dust around the Kerr black hole. Here we will discuss general situation, specially for above three situations.

The black hole horizon is determined by the following equation

$$\Delta_r = r^2 - 2Mr + a^2 - \alpha r^{1-3\omega} = 0. \quad (3)$$

This equation has three horizons if $-1 < \omega < -1/3$, including the horizon, outer horizon and cosmological horizon. In the case of $\omega > -1/3$, the cosmological horizon will vanish and the equation (3) exists two horizons. Here we only consider the inner and outer horizons. The cosmological horizon does not exist, but this situation is also interesting because its metric describes other energy matters such as radiation and dust around the Kerr black hole. Because α is small, we make the perturbation method for α in horizon equation ([S. Mahamat et al. \(2011,?\)](#)). In this paper we make the quantities relating to α one-order, the horizon is written as

$$r_{\pm} = R_{\pm} + \epsilon_{\pm}, \quad (4)$$

where R_{\pm} is the horizon of Kerr black hole. r_+ is the outer horizon or event horizon \mathcal{H}^+ and r_- is the inner horizon or Cauchy horizon \mathcal{H}^- . Substituting (4) into (3), we expand the equation to the first order term. The equation becomes

$$(2R_{\pm} - 2M - \alpha(1 - 3\omega)R_{\pm}^{-3\omega})\epsilon_{\pm} - \alpha R_{\pm}^{1-3\omega} = 0. \quad (5)$$

The solution of equation (5) satisfies the condition of $\epsilon_{\pm} = 0$ for $\alpha = 0$, therefore the solution is given by

$$\epsilon_{\pm} = \frac{\alpha R_{\pm}^{1-3\omega}}{2R_{\pm} - 2M - \alpha(1-3\omega)R_{\pm}^{-3\omega}} \simeq \pm \frac{\alpha(M \pm \sqrt{M^2 - a^2})^{1-3\omega}}{2\sqrt{M^2 - a^2}},$$

$$R_{\pm} = M \pm \sqrt{M^2 - a^2}, \quad (6)$$

and

$$r_+ \simeq M + \sqrt{M^2 - a^2} + \frac{\alpha(M + \sqrt{M^2 - a^2})^{1-3\omega}}{2\sqrt{M^2 - a^2} - \alpha(1-3\omega)(M + \sqrt{M^2 - a^2})^{-3\omega}}$$

$$\simeq M + \sqrt{M^2 - a^2} + \frac{\alpha(M + \sqrt{M^2 - a^2})^{1-3\omega}}{2\sqrt{M^2 - a^2}}, \quad (7)$$

$$r_- \simeq M - \sqrt{M^2 - a^2} + \frac{\alpha(M - \sqrt{M^2 - a^2})^{1-3\omega}}{-2\sqrt{M^2 - a^2} - \alpha(1-3\omega)(M - \sqrt{M^2 - a^2})^{-3\omega}}$$

$$\simeq M - \sqrt{M^2 - a^2} - \frac{\alpha(M - \sqrt{M^2 - a^2})^{1-3\omega}}{2\sqrt{M^2 - a^2}}. \quad (8)$$

Here we use three special examples with different ω to show the horizon and product of horizon. The product of horizon for ω is given by

$$r_+ r_- = a^2 + \frac{\alpha a^2}{2\sqrt{M^2 - a^2}}(R_+^{-3\omega} - (R_-^{-3\omega})). \quad (9)$$

Case one: $\omega = -2/3$ corresponding to quintessence, there is no exact formalism horizon for $\Delta_r = 0$. From equation (2) and (2), we find

$$r_{\pm} \simeq M \pm \sqrt{M^2 - a^2} \pm \frac{\alpha(M \pm \sqrt{M^2 - a^2})^3}{2\sqrt{M^2 - a^2}}, \quad (10)$$

and the product of the horizon is given by

$$r_+ r_- = a^2 + 2Ma^2\alpha. \quad (11)$$

Case two: $\omega = 1/3$ representing to radiation, there are two exact horizon expressions for $\Delta_r = 0$. From equation (3), we find that

$$r_{\pm} = M \pm \sqrt{M^2 - a^2 + \alpha}, \quad (12)$$

and the product of the horizon is given by

$$r_+ r_- = a^2 - \alpha. \quad (13)$$

Case three: $\omega = 0$ relating to dust, there is exact formalism for two horizons. From equation (3), we find

$$r_{\pm} = M + \frac{\alpha}{2} \pm \sqrt{M^2 - a^2 + M\alpha + \frac{\alpha^2}{4}}, \quad (14)$$

and the product of the horizon is given by

$$r_+ r_- = a^2. \quad (15)$$

In summary, the product of horizon is determined by ω and α . For the quintessence around Kerr black hole, the product of horizon is mass dependent. When $\alpha = 0$, the result is opposite. For the radiation and dust around Kerr black hole, the products of horizons are independent on mass, but depend on spin and α .

For stationary and axial symmetric black hole, the areas of the inner and outer horizons of the black hole are given by

$$\begin{aligned} A_{\pm} &= \int \sqrt{g} d\theta d\phi = \int_0^{2\pi} \int_0^{\pi} \sqrt{g_{\theta\theta} g_{\phi\phi}} d\theta d\phi = 4\pi(r_{\pm}^2 + a^2) \simeq 4\pi(R_{\pm}^2 + 2R_{\pm}\epsilon_{\pm} + a^2) \\ &= 4\pi(2M^2 \pm 2M\sqrt{M^2 - a^2}) \pm 4\pi \frac{\alpha(M \pm \sqrt{M^2 - a^2})^{2-3\omega}}{\sqrt{M^2 - a^2}}. \end{aligned} \quad (16)$$

The semi-classical Bekenstein-Hawking entropy at \mathcal{H}^{\pm} is given by (Hawking (1971))

$$S_{\pm} = \frac{A_{\pm}}{4} \simeq \pi(2M^2 \pm 2M\sqrt{M^2 - a^2}) \pm \pi \frac{\alpha(M \pm \sqrt{M^2 - a^2})^{2-3\omega}}{\sqrt{M^2 - a^2}}. \quad (17)$$

The surface gravity of the black hole is defined by

$$\begin{aligned} \kappa_{\pm} &= \lim_{r \rightarrow r_{\pm}} \left(-\frac{1}{2} \sqrt{\frac{g^{rr}}{-g'_{tt}}} g'_{tt,r} \right) = \frac{r_+ - r_-}{2(r_{\pm}^2 + a^2)} = \frac{R_+ - R_- + \epsilon_+ - \epsilon_-}{2((R_{\pm} + \epsilon_{\pm})^2 + a^2)} \\ &\simeq \frac{\sqrt{M^2 - a^2}}{2MR_{\pm}} + \frac{\alpha}{4MR_{\pm}\sqrt{M^2 - a^2}} \left[\frac{1}{2}(R_+^{1-3\omega} + R_-^{1-3\omega}) \mp \frac{\sqrt{M^2 - a^2} R_{\pm}^{1-3\omega}}{M} \right], \end{aligned} \quad (18)$$

where $g'_{tt} = g_{tt} - g_{t\phi}^2/g_{\phi\phi}$.

From the black hole thermodynamics, the Hawking temperature for \mathcal{H}^{\pm} is

$$\begin{aligned} T_{\pm} &= \frac{\kappa_{\pm}}{2\pi} = \frac{R_+ - R_- + \epsilon_+ - \epsilon_-}{4\pi((R_{\pm} + \epsilon_{\pm})^2 + a^2)} \\ &\simeq \frac{\sqrt{M^2 - a^2}}{4\pi MR_{\pm}} + \frac{\alpha}{8\pi MR_{\pm}\sqrt{M^2 - a^2}} \left[\frac{1}{2}(R_+^{1-3\omega} + R_-^{1-3\omega}) \mp \frac{\sqrt{M^2 - a^2} R_{\pm}^{1-3\omega}}{M} \right]. \end{aligned} \quad (19)$$

For rotation black hole, the rotation velocity of the horizon is

$$\Omega_{\pm} = \lim_{r \rightarrow r_{\pm}} \left(\frac{-g_{t\phi}}{g_{\phi\phi}} \right) = \frac{a}{r_{\pm}^2 + a^2}. \quad (20)$$

Using $A_{\pm}, S_{\pm}, T_{\pm}$ and κ_{\pm} , we can calculate the Komar energy (Komar (1959)), the product of surface gravity, the surface temperatures, the areas and entropy at \mathcal{H}^{\pm} , which are given by

the Komar energy

$$\begin{aligned} E_{\pm} &= 2S_{\pm}T_{\pm} = \frac{(R_{\pm}^2 + 2R_{\pm}\epsilon_{\pm})(R_{+} - R_{-} + \epsilon_{+} - \epsilon_{-})}{2((R_{\pm} + \epsilon_{\pm})^2 + a^2)} \\ &\simeq \frac{R_{\pm}\sqrt{M^2 - a^2}}{2M} + \frac{R_{\pm}\alpha}{2M} \left[\frac{1}{2\sqrt{M^2 - a^2}} \left(\frac{1}{2}(R_{+}^{1-3\omega} + R_{-}^{1-3\omega}) \mp \frac{\sqrt{M^2 - a^2}R_{\pm}^{1-3\omega}}{M} \right) \pm R_{\pm}^{1-3\omega} \right], \end{aligned} \quad (21)$$

the product of surface gravity and surface temperature

$$\begin{aligned} \kappa_{+}\kappa_{-} &= 4\pi^2 T_{+}T_{-} = \frac{(R_{+} - R_{-} + \epsilon_{+} - \epsilon_{-})^2}{2((R_{+} + \epsilon_{+})^2 + a^2)(R_{-} + \epsilon_{-})^2 + a^2)} \\ &\simeq \frac{M^2 - a^2}{4M^2a^2} + \frac{\alpha}{8M^2a^2} \left[R_{+}^{1-3\omega} + R_{-}^{1-3\omega} + \frac{\sqrt{M^2 - a^2}}{M} (R_{-}^{1-3\omega} - R_{+}^{1-3\omega}) \right], \end{aligned} \quad (22)$$

the product of Komar energy

$$\begin{aligned} E_{+}E_{-} &= \frac{(R_{+}^2 + 2R_{+}\epsilon_{+})(R_{-}^2 + 2R_{-}\epsilon_{-})(R_{+} - R_{-} + \epsilon_{+} - \epsilon_{-})}{2((R_{+} + \epsilon_{+})^2 + a^2)(R_{-} + \epsilon_{-})^2 + a^2)} \\ &\simeq \frac{a^2(M^2 - a^2)}{4M^2} + \frac{a^2\alpha\sqrt{M^2 - a^2}}{4M^2} \left[\frac{1}{2\sqrt{M^2 - a^2}} (R_{+}^{1-3\omega} + R_{-}^{1-3\omega}) + \left(1 - \frac{1}{2M}\right)(R_{+}^{1-3\omega} - R_{-}^{1-3\omega}) \right], \end{aligned} \quad (23)$$

the product of areas and entropy

$$\begin{aligned} A_{+}A_{-} &= 16S_{+}S_{-} = 16\pi^2 (R_{+}^2 + 2R_{+}\epsilon_{+})(R_{-}^2 + 2R_{-}\epsilon_{-}) \\ &\simeq a^4 + \frac{\alpha a^4}{\sqrt{M^2 - a^2}} (R_{+}^{-3\omega} - R_{-}^{-3\omega}). \end{aligned} \quad (24)$$

From the equation (2), (2) and (2), we find that the products of surface gravity, the surface temperatures and the Komar energy are related to the mass of black hole for different ω . This property is the same with that of spherically symmetric black hole ($a = 0$). The relation between the products of area (entropy) and the mass is determined by ω and α . When $\alpha = 0$,

the products of area and entropy are independent on mass. Here we give the results for three cases: quintessence, radiation and dust.

Case one: the quintessence with $\omega = -2/3$, the products become

$$\kappa_+ \kappa_- = 4\pi^2 T_+ T_- \simeq \frac{M^2 - a^2}{4M^2 a^2} + \alpha \left(\frac{1}{2M} - \frac{a^2}{4M^3} \right), \quad (25)$$

$$E_+ E_- \simeq \frac{a^2(M^2 - a^2)}{4M^2} + \frac{a^2 \alpha \sqrt{M^2 - a^2}}{4M^2} ((4M^2 - a^2)(4M^3 - 4Ma^2 + 2a^2) - 4M^2 a^2), \quad (26)$$

$$A_+ A_- \simeq 16S_+ S_- = a^4 + 4Ma^4 \alpha. \quad (27)$$

Case two: the radiation with $\omega = 1/3$, the products become

$$\kappa_+ \kappa_- = 4\pi^2 T_+ T_- = \frac{M^2 - a^2}{4M^2 a^2} + \frac{\alpha}{4M^2 a^2}, \quad (28)$$

$$E_+ E_- = \frac{a^2(M^2 - a^2)}{4M^2} + \frac{a^2 \alpha}{4M^2}, \quad (29)$$

$$A_+ A_- = 16S_+ S_- = a^4 - 2Ma^2 \alpha. \quad (30)$$

Case three: the dust with $\omega = 0$, the products are

$$\kappa_+ \kappa_- = 4\pi^2 T_+ T_- = \frac{M^2 - a^2}{4M^2 a^2} + \frac{\alpha}{4M^3}, \quad (31)$$

$$E_+ E_- = \frac{a^2(M^2 - a^2)}{4M^2} + \frac{a^2 \alpha}{4M^3} (2M(M^2 - a^2) + a^2), \quad (32)$$

$$A_+ A_- = 16S_+ S_- = a^4. \quad (33)$$

2.1. Generalized Smarr Formula

From the formula (15), the general formalism (Smarr (1973,?)) relating to the entropy, angular momentum and mass of the black hole for any ω is

$$M = \frac{1}{2} \sqrt{\frac{S}{\pi} - \frac{J^2}{M^2}} + \frac{J^2}{2M^2} \sqrt{\frac{\pi M^2}{SM^2 - \pi J^2}} - \frac{\alpha}{2} \left(\frac{S}{\pi} - \frac{J^2}{M^2} \right)^{-\frac{3\omega}{2}}. \quad (34)$$

Because the equations are very complex for general ω , we don't get the mass formula. But for special ω , we will calculate the mass formula and derive the state equation. Here we will calculate the special case: the radiation with $\omega = 1/3$ around Kerr black hole. For simple, we only consider the property on the outer horizon. The event horizon has exact formalism, and the equation (34) becomes

$$S = \pi(2M^2 + 2\sqrt{M^4 - J^2 + M^2\alpha} + \alpha). \quad (35)$$

We then get the function of $M = M(S, J, \alpha)$ as

$$M^2 = \frac{S}{4\pi} - \frac{\alpha}{2} + \frac{\pi\alpha^2}{4S} + \frac{\pi J^2}{S}. \quad (36)$$

We define the quantities relating to S, J and α , the effective surface tension and angular velocity are given by

$$T = \left(\frac{\partial M}{\partial S} \right)_J = \frac{\frac{1}{2\pi} - \frac{\pi\alpha^2}{8S^2} - \frac{\pi J^2}{2S^2}}{\sqrt{\frac{S}{4\pi} - \frac{\alpha}{2} + \frac{\pi\alpha^2}{4S} + \frac{\pi J^2}{S}}}, \quad (37)$$

$$\Omega = \left(\frac{\partial M}{\partial J} \right)_S = \frac{\pi J}{MS}. \quad (38)$$

According to the first law of thermodynamics, the mass change of black hole is determined by the areas and angular momentum as

$$dM = TdA + \Omega dJ, \quad (39)$$

where T is effective surface tension given by

$$\begin{aligned} T_{\pm} &= \frac{1}{M} \left(\frac{1}{2\pi} - \frac{\pi\alpha^2}{8S^2} - \frac{\pi J^2}{2S^2} \right) \\ &= \frac{1}{M} \left(\frac{1}{2\pi} - \left(\frac{\pi\alpha^2}{8} - \frac{\pi M^2 a^2}{2} \right) \frac{1}{\pi^2 (2M^2 \pm 2M\sqrt{M^2 - a^2 + \alpha} + \alpha)^2} \right) \end{aligned}$$

$$= \frac{1}{8\pi} \frac{\sqrt{M^2 - a^2 + \alpha}}{2M^2 \pm 2M\sqrt{M^2 - a^2 + \alpha} + \alpha} = \frac{\kappa_{\pm}}{8\pi}, \quad (40)$$

$S = \pi(2M^2 \pm 2M\sqrt{M^2 - a^2 + \alpha} + \alpha)$ and $J = Ma$.

The angular velocity Ω is given by

$$\begin{aligned} \Omega &= \frac{\pi J}{MS} \\ &= \frac{\pi Ma}{M\pi(2M^2 \pm 2(M^4 - J^2)^{1/2})} \\ &= \frac{a}{2Mr_{\pm}} = \frac{a}{r_{\pm}^2 + a^2} = \Omega_{\pm}. \end{aligned} \quad (41)$$

2.2. Christodoulou-Ruffini Mass Formula

The black hole mass can decrease or increase, but the irreducible mass M_{irr} of black hole can not decrease (Christodoulou (1970); Christodoulou & Ruffini (1971)). The increase of M_{irr} will lead to many physical processes in black hole. According to Hawking, the black hole satisfies the thermodynamics second law. The areas of horizon or entropy all increase, e.g., $dS = dA/4 \geq 0$, leading a relation between horizon area and irreducible mass for black hole.

In general quintessence, the black hole has regular event horizon and Cauchy horizons, thus the irreducible mass and horizon areas satisfy the following equation

$$\begin{aligned} M_{irr\pm} &= \sqrt{\frac{A_{\pm}}{16\pi}} = \sqrt{\frac{r_{\pm}^2 + a^2}{4}} \\ &= \left(\frac{1}{2}M^2 \pm \frac{1}{2}M\sqrt{M^2 - a^2} \pm \frac{\alpha(M \pm \sqrt{M^2 - a^2})^{2-3\omega}}{4\sqrt{M^2 - a^2}} \right)^{1/2}, \end{aligned} \quad (42)$$

where M_{irr+} and M_{irr-} are black hole irreducible masses on the outer and inner horizons. The product of the irreducible masses is given by

$$\begin{aligned} M_{irr+}M_{irr-} &= \frac{\sqrt{A_+A_-}}{16\pi} \\ &= \frac{\sqrt{a^4 + \frac{\alpha a^4}{\sqrt{M^2 - a^2}}(R_+^{-3\omega} - R_-^{-3\omega})}}{16\pi}. \end{aligned} \quad (43)$$

The rest mass of rotating black hole is given by the following expression

$$M^2 = M_{irr\pm}^2 + \frac{J^2}{4M_{irr\pm}^2} \quad (44)$$

3. PHASE TRANSITION

In this section, we study the phase transition in these black holes. The heat capacity of the black hole is an interesting thermodynamical property presenting the stability and instability of the black hole. General black hole has a negative heat capacity to be unstable and produce Hawking radiation (Gross et al. (1982)). But if the black hole brings angular momentum or charge, the heat capacity could be positive, and the phase transition will occur. (Davies (1977)).

For the quintessence matter (other energy matter) around spherical symmetry black hole, the phase transition has been discussed (Majeed et al. (2015); Tharanath et al. (2014); Thomas et al. (2012); S. Mahamat et al. (2011); Ghaderi & Malakolkalami (2016); Wei & Ren (2013)). For rotational Kiselev black hole or Kerr-Newman black hole, we want to analyze the phase transition. In order to obtain the equation of state for thermodynamical system. Substituting (15) into (3), we write the black hole mass as a function of black hole entropy and angular momentum $M = M(S, J, \alpha, \omega)$ given by

$$M = \frac{1}{2} \sqrt{\frac{S}{\pi} - \frac{J^2}{M^2}} + \frac{J^2}{2M^2} \sqrt{\frac{\pi M^2}{SM^2 - \pi J^2}} - \frac{\alpha}{2} \left(\frac{S}{\pi} - \frac{J^2}{M^2} \right)^{-\frac{3\omega}{2}}. \quad (45)$$

From the section 2, we know that the first law of thermodynamics for these black holes is given by

$$dM = TdS + \Omega dJ. \quad (46)$$

Using the relation (45), we define other thermodynamical quantities given by

$$T = \left(\frac{\partial M}{\partial S} \right)_J = \frac{\frac{1}{4\pi} + \frac{3\alpha}{4\pi} \omega \left(\frac{S}{\pi} - \frac{J^2}{M^2} \right)^{-\frac{3}{2}\omega - \frac{1}{2}}}{\sqrt{\frac{S}{\pi} - \frac{J^2}{M^2}} - \frac{J^2}{2M^3} + \frac{\pi J^2 (4M + J^2)}{4M^2 (SM^2 - \pi J^2)} - \frac{3\alpha J^2}{2M^3} \omega \left(\frac{S}{\pi} - \frac{J^2}{M^2} \right)^{-\frac{3}{2}\omega - \frac{1}{2}}}, \quad (47)$$

$$\Omega = \left(\frac{\partial M}{\partial J} \right)_S =$$

$$\frac{-\frac{J}{2M^2} + \frac{\pi J}{SM^2 - \pi J^2} + \frac{\pi^2 J^3}{2(SM^2 - \pi J^2)^2} - \frac{3\alpha}{2}\omega \frac{J}{M^2} \left(\frac{S}{\pi} - \frac{J^2}{M^2}\right)^{-\frac{3}{2}\omega - \frac{1}{2}}}{\sqrt{\frac{S}{\pi} - \frac{J^2}{M^2} - \frac{J^2}{2M^3} + \frac{\pi J^2}{M(SM^2 - \pi J^2)} + \frac{\pi^2 J^4}{2M(SM^2 - \pi J^2)^2} - \frac{3\alpha\omega J^2}{2M^3} \left(\frac{S}{\pi} - \frac{J^2}{M^2}\right)^{-\frac{3}{2}\omega - \frac{1}{2}}}.$$
(48)

In order to simplify the equation, we set

$$T = \frac{X}{Y} \quad X = \frac{1}{4\pi} + \frac{3\alpha}{4\pi}\omega \left(\frac{S}{\pi} - \frac{J^2}{M^2}\right)^{-\frac{3}{2}\omega - \frac{1}{2}},$$

$$Y = \sqrt{\frac{S}{\pi} - \frac{J^2}{M^2} - \frac{J^2}{2M^3} + \frac{\pi J^2(4M + J^2)}{4M^2(SM^2 - \pi J^2)} - \frac{3\alpha J^2}{2M^3}\omega \left(\frac{S}{\pi} - \frac{J^2}{M^2}\right)^{-\frac{3}{2}\omega - \frac{1}{2}}}, \quad (49)$$

$$C_J = T \left(\frac{\partial S}{\partial T}\right)_J = \frac{XY}{\frac{\partial X}{\partial S}Y - X\frac{\partial Y}{\partial S}}, \quad (50)$$

where $\frac{\partial X}{\partial S}Y - X\frac{\partial Y}{\partial S}$ is given by

$$\begin{aligned} \frac{\partial X}{\partial S}Y - X\frac{\partial Y}{\partial S} = & -\frac{3\alpha\omega}{8\pi}(3\omega + 1)\left(\frac{S}{\pi} - \frac{J^2}{M^2}\right)^{-\frac{3}{2}(\omega+1)}\left(\frac{1}{\pi} + \frac{2J^2}{M^3}T\right) - \frac{1}{4\pi}\left(1 + 3\alpha\omega\left(\frac{S}{\pi} - \frac{J^2}{M^2}\right)^{-\frac{3}{2}(\omega+1)}\right) \\ & \left[\frac{\frac{1}{\pi} + \frac{2J^2}{M^3}T}{2\sqrt{\frac{S}{\pi} - \frac{J^2}{M^2}}} + \frac{3J^2}{2M^4}T - \frac{\pi J^2(4M + J^2)}{4(SM^2 - \pi J^2)^2} - \right. \\ & \left. \frac{2\pi J^2((SM^2 - \pi J^2)(2M^2 + MJ^2) + SM^3(4M + J^2))T}{4M^4(SM^2 - \pi J^2)^2} + \frac{3\alpha J^2\omega}{4M^3}(3\omega + 1)\left(\frac{S}{\pi} - \frac{J^2}{M^2}\right)^{-\frac{3}{2}(\omega+1)}\right. \\ & \left. \left(\frac{1}{\pi} + \frac{2J^2}{M^3}T\right) + \frac{9\alpha J^2}{2M^4}\omega \left(\frac{S}{\pi} - \frac{J^2}{M^2}\right)^{-\frac{3}{2}\omega - \frac{1}{2}}T\right]. \end{aligned} \quad (51)$$

The phase transition will produce when the following condition satisfies

$$\frac{\partial X}{\partial S}Y = X\frac{\partial Y}{\partial S}. \quad (52)$$

For general ω , we can not give the expression of C_J . Here we take $\omega = 1/3$ to study the feature of phase transition. The generalized Smarr mass function has been obtained in section 2.1 as

$$M^2 = \frac{S}{4\pi} - \frac{\alpha}{2} + \frac{\pi\alpha^2}{4S} + \frac{\pi J^2}{S}. \quad (53)$$

Using this equation, we get the expression for T, Ω, C_J as

$$T = \frac{\frac{1}{4\pi} - \frac{\pi\alpha^2}{4S^2} - \frac{\pi J^2}{S^2}}{2\sqrt{\frac{S}{4\pi} - \frac{\alpha}{2} + \frac{\pi\alpha^2}{4S} + \frac{\pi J^2}{S}}}, \quad (54)$$

$$\Omega = \frac{\pi J}{MS} = \frac{\pi J}{S\sqrt{\frac{S}{4\pi} - \frac{\alpha}{2} + \frac{\pi\alpha^2}{4S} + \frac{\pi J^2}{S}}}, \quad (55)$$

$$C_J = \frac{2(\frac{S}{4\pi} - \frac{\alpha}{2} + \frac{\pi\alpha^2}{4S} + \frac{\pi J^2}{S})(\frac{1}{4\pi} - \frac{\pi\alpha^2}{4S^2} - \frac{\pi J^2}{S^2})}{(\frac{\pi\alpha^2}{S^3} + \frac{4\pi J^2}{S^3})(\frac{S}{4\pi} - \frac{\alpha}{2} + \frac{\pi\alpha^2}{4S} + \frac{\pi J^2}{S}) - (\frac{1}{4\pi} - \frac{\pi\alpha^2}{4S^2} - \frac{\pi J^2}{S^2})^2}. \quad (56)$$

The phase transition will produce when the following equation satisfies

$$(\frac{\pi\alpha^2}{S^3} + \frac{4\pi J^2}{S^3})(\frac{S}{4\pi} - \frac{\alpha}{2} + \frac{\pi\alpha^2}{4S} + \frac{\pi J^2}{S}) = (\frac{1}{4\pi} - \frac{\pi\alpha^2}{4S^2} - \frac{\pi J^2}{S^2})^2. \quad (57)$$

In order to analyze the critical point of the phase transition, we set $M = 1$ to obtain $J = a$ and $S = \pi(2 + \alpha + 2\sqrt{1 + \alpha - a^2})$. The equation becomes

$$(2 + \alpha + 2\sqrt{1 + \alpha - a^2})(\alpha^2 + 4a^2) = (a^2 + \frac{\alpha^2}{4} - \frac{1}{4}(8 + 8\alpha + \alpha^2 - 4a^2 + 4(2 + \alpha)\sqrt{1 + \alpha - a^2}))^2. \quad (58)$$

Through numerical calculation, we find that the phase transition with $a \leq 1$ demands $\alpha \leq 1$. For different α , the critical black hole spin a is given in Table 1. We find that the existence of other energy matters will boost the critical black hole spin a . When $\alpha = 0$ it reduces to Kerr black hole case and the phase transition is second order phase transition (Davies (1977)). When $a = 0$ it reduces to spherically symmetric case and the phase transition is also second order phase transition (Ghaderi & Malakolkalami (2016); Majeed et al. (2015); Tharanath et al. (2014)). For the rotational black hole in radiation case the phase transition is second order phase transition.

Table 1: Variation of the parameter α versus black hole spin a (M=1)

parameter α	0.0000	0.0010	0.0100	0.2000	0.5000	0.8000	0.9500	1
black hole spin a	0.6813	0.6817	0.6855	0.7601	0.8612	0.9480	0.9873	1

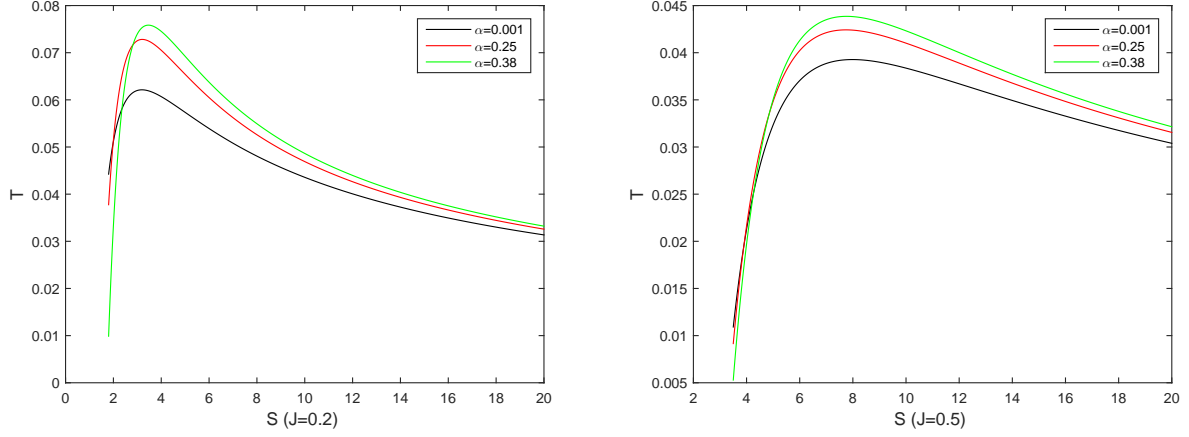


Fig. 1.— Semi-classical Hawking temperature T versus entropy S for different angular momentum ($J=0.2, J=0.5, \omega = 1/3$)

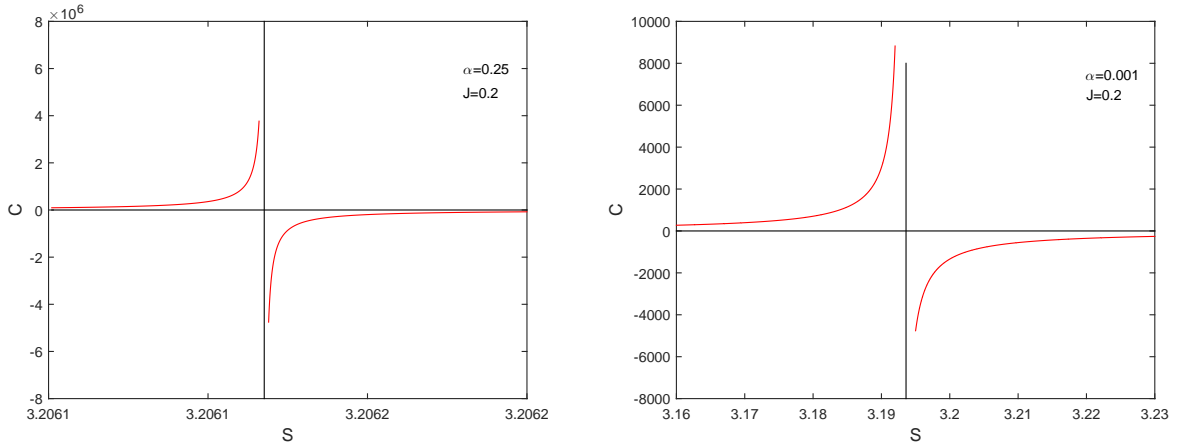


Fig. 2.— Variation of the heat capacity C_J versus entropy S at constant angular momentum for different parameter α . ($\omega = 1/3$)

We plot the variation of the semi-classical Hawking temperature T versus the entropy S for different parameters α and angular momentum J in picture 1. The variation of the heat capacity C_J versus the entropy S for different parameters α is shown in picture 2. From picture 1, we find that when α decreases the semi-classical hawking temperature will also decrease. From picture 2, we find that the heat capacity will pass a critical point with the entropy, representing the phase transition point. Interestingly, the critical point shifts to higher entropy when the parameter α increases.

4. SUMMARY

We have discussed some thermodynamical features on two horizons for rotation Kiselev black hole. By horizon perturbation, we obtain black hole area, surface gravity, entropy and so on. Through calculating the products of these quantities, we find that they depend on the parameters ω and α , and are different from ones in Kerr Black Hole. We also generalize the Smarr mass formula and study the Christodoulou-Ruffini mass formula, and find that the first law of thermodynamics is also satisfied. Finally we calculate the phase transition for the black hole with $\omega = 1/3$, and find that the phase transition point shifts to higher entropy when the parameter α increases.

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